Workshop MODNUM

Technical meeting at AUB

New Boundary Conditions for a Reduced Domain having a Point Symmetry

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Motivation(1)

Heat transfer in porous media has many applications:

- Arphymat Program:study of archaeological hearths.
- Agriculture drought
- Industrial Drying (wood, concrete walls, food ...)



www.4to40.com/science/print.asp?p=Discovery_of_Fire

http://www.recirculatingfarms.org/recirculating-farming-one-way-to-beat-a-drought

tinyhttp://en.wikipedia.org/wiki/File:Dry_fruit.jpg

Motivation(2)

Importance of numerical simulation:

- No time or space limitations.
- Can treat complex problems
- less expensive than experimental studies.

Limitation of Numerical Simulation lies in the lack of information about medium properties:

* Effective thermal conductivity of granular medium : not easy to calculate in unsaturated medium.

Motivation(3)

Effective thermal conductivity of a complex medium.

- Fourier law: $\Phi = -\lambda_e . \frac{\Delta T}{\Delta y}$ (Fourier law is a local equation but here we adopt it from a global point of view).
- Need to impose ΔT and Δy
- Calculate Φ.
- Deduce λ_e.

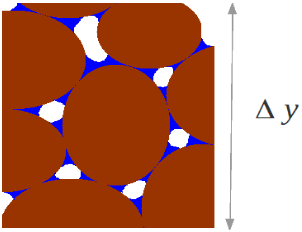
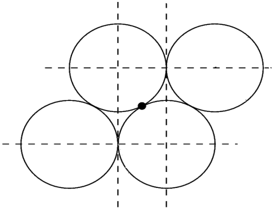
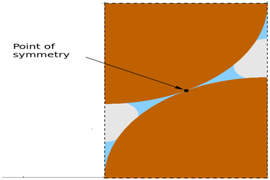


Figure: Example of heterogenous complex porous medium (solid, liquid and gas).

Introduction (1)

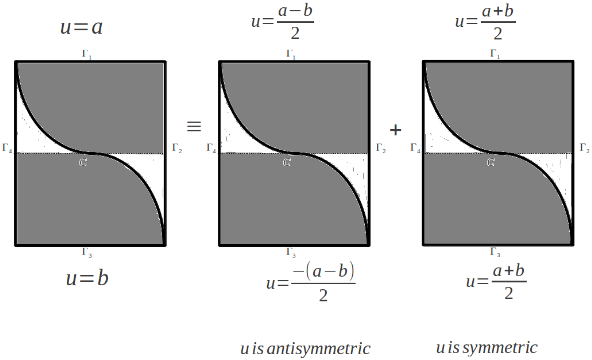
- Reducing the computational cost of algorithms.
- Symmetries and periodicities can help in reducing the computational cost.
- A domain symmetric with respect to a point.
- How to split further the domain?
- Boundary Conditions to be imposed?





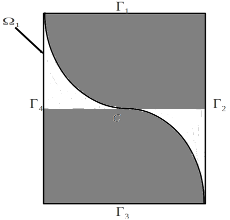
Introduction (2)

- We are concerned in solving the steady state heat elliptic PDE: $\mathrm{div}(\lambda(\mathbf{x}).\nabla u) = 0, \quad \lambda(\mathbf{x}) > 0$



Introduction (2)

- We are concerned in solving the steady state heat elliptic PDE illustrated by system 1 over the domain Ω_1 represented by figure 1 below where $u = 0$ on the point C :

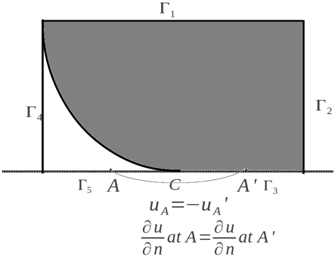


$$\begin{cases} \mathrm{div}(\lambda(\mathbf{x}).\nabla u) = 0, \quad \lambda(\mathbf{x}) > 0, \\ u = 1 \text{ on } \Gamma_1, \\ u = -1 \text{ on } \Gamma_3, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \Gamma_2 \cup \Gamma_4, \end{cases} \tag{1}$$

The geometry of the domain and $\lambda(\mathbf{x})$ are both symmetric with respect to the point C.

Introduction (3)

- We proved that the solution of system (1) over Ω_1 is anti-symmetric.
- We proved the existence and uniqueness of solution of system (2) over the domain Ω_2 represented below where $u_C = 0$:



$$\begin{cases} \text{-div}(\lambda(\mathbf{x}).\nabla u) = 0, \\ u = 1 \text{ on } \mathit{\Gamma}_1, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \mathit{\Gamma}_2 \cup \mathit{\Gamma}_4 \\ u|_{\mathit{\Gamma}_3} \text{ at A } = -u|_{\mathit{\Gamma}_5} \text{ at A' }, \\ \dfrac{\partial u}{\partial n}|_{\mathit{\Gamma}_3} \text{ at A } = \dfrac{\partial u}{\partial n}|_{\mathit{\Gamma}_5} \text{ at A' }. \end{cases} \quad (2)$$

- Numerical Application and Numerical Results.

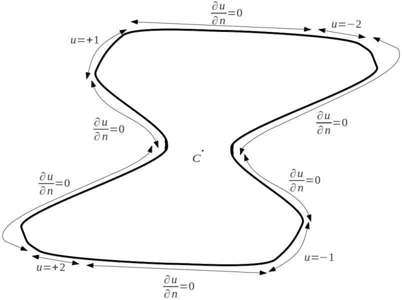
1
Demonstration of the Anti-symmetry Property of the Solution

Problem 1:

$$\begin{cases} \operatorname{div}(\lambda(\mathbf{x}).\nabla u) = 0, \quad \lambda(\mathbf{x}) > 0, \\ u = g \text{ on } \Gamma_D, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N. \end{cases}$$

- $\lambda(\mathbf{x})$ is symmetric with respect to the point C.
- g is antisymmetric with respect to C.

$\rightarrow$ Solution u_1 of problem 1 exists and unique.



Problem 2 :

$$\begin{cases} \operatorname{div}(\lambda(\mathbf{x}).\nabla u) = 0, \quad \lambda(\mathbf{x}) > 0, \\ u = -g \text{ on } \Gamma_D, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N, \end{cases}$$

→ $u_2 = -u_1$ is the solution of problem 2.(unique)

Problem 3 :

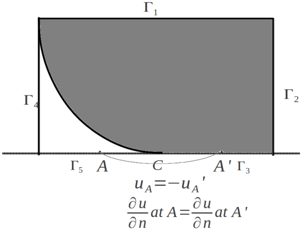
- Ω is rotated by 180°
- Boundary conditions were kept as they were in problem 1
- Problem 3 will be equivalent to Problem 2.

→ $u_3 = -u_1$ is the solution of Problem 3.

→ u is antisymmetric with respect to point C.

2 Existence and Uniqueness of the Solution of the new problem

We used the Lax Milgram theorem to prove the existence and uniqueness of system 3 over Ω_2 represented by figure 2 where $u_C = 0$



$$\begin{cases} \text{-div}(\lambda(\mathbf{x}).\nabla u) = 0, \\ u = 1 \text{ on } \Gamma_1, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \Gamma_2 \cup \Gamma_4 \\ u|_{\Gamma_3} \text{ at A } = -u|_{\Gamma_5} \text{ at A' }, \\ \dfrac{\partial u}{\partial n}|_{\Gamma_3} \text{ at A } = \dfrac{\partial u}{\partial n}|_{\Gamma_5} \text{ at A' }. \end{cases} \quad (3)$$

Let $\Gamma_D = \Gamma_1 \bigcup C$

$$g_D = \begin{cases} 1 & on & \Gamma_1 \\ 0 & on & C \end{cases}$$

Also, use $u = U + G$ where G is a Dirichlet lift. ($G \equiv g_D$ on Γ_D)

- We get the system 4 below:

$$\begin{cases} \text{-div}(\lambda(\mathbf{x}).\nabla U) = \text{div}(\lambda(\mathbf{x}).\nabla G) \text{ in } \Omega, \\ U = 0 \text{ on } \Gamma_D, \\ \dfrac{\partial(U+G)}{\partial n} = 0 \text{ on } \Gamma_2 \cup \Gamma_4 \\ (U+G)|_{\Gamma_3} \text{ at A } = -(U+G)|_{\Gamma_5} \text{ at A'} , \\ \dfrac{\partial(U+G)}{\partial n}|_{\Gamma_3} \text{ at A } = \dfrac{\partial(U+G)}{\partial n}|_{\Gamma_5} \text{ at A'} , \end{cases} \tag{4}$$

- Using Green's identity, we end up with :

$$\int_\Omega (\lambda.\nabla U).\nabla\phi.\partial\mathbf{x} = -\int_\Omega (\lambda.\nabla G).\nabla\phi.\partial\mathbf{x}$$

- The weak formulation of the problem is to find $U \in H$ such that:

$$a(U,\phi) = l(\phi)$$

$$\begin{cases} a(U,\phi) : V \times V \to \mathbb{R} \\ \text{where } a(U,\phi) = \int_\Omega (\lambda.\nabla U).\nabla\phi.\partial\mathbf{x} \\ l(\phi) : V \to \mathbb{R} \\ \text{where } l(\phi) = -\int_\Omega (\lambda.\nabla G).\nabla\phi.\partial\mathbf{x} \end{cases}$$

  $a(U,\phi)$ is bilinear, bounded and coercive.
- According to the Lax-Milgram theorem, there exists a unique solution $U \in H$ to the equation $a(U,\phi) = l(\phi)$
- The final solution is $u = U + G$ which satisfies both the Dirichlet and Neumann Boundary Conditions.

3
Numerical Application

- A simple geometric representation of the 2D problem is illustrated by the figures and system below where $u_C = 0$:

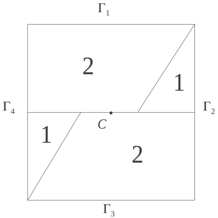


Figure: Geometrical representation of the square shape computational domain.

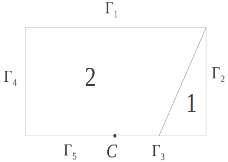


Figure: Geometrical representation of the reduced computational domain.

$$\begin{cases} \text{-div}(\lambda(\mathbf{x}).\nabla u) = 0, \\ u = 1 \text{ on } \mathit{\Gamma}_1, \\ u = -1 \text{ on } \mathit{\Gamma}_3, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \mathit{\Gamma}_2 \cup \mathit{\Gamma}_4 \end{cases} \tag{5}$$

Numerical Method

- We used the Mixed Finite Element Method (MFEM) because the dual variable (flux here) is computed as a fundamental unknown.
- Using Fourier law $\mathbf{q} = -\lambda(\mathbf{x}).\nabla u$ and conservation law $\sum q = 0$ on each cell , we get a system of equations with uknowns u_i :

$$\begin{cases} a_{1,1}u_1 + a_{1,2}u_2 + a_{1,n+1}u_{n+1} = b_1 \\ \quad\quad\quad\quad . \\ \quad\quad\quad\quad . \\ a_{i,i-n}u_{i-n} + a_{i,i-1}u_{i-1} + a_{i,i}u_i + a_{i,i+1}u_{i+1} + a_{i,i+n}u_{i+n} = b_i \\ \quad\quad\quad\quad . \\ \quad\quad\quad\quad . \\ a_{N,N-n}u_{N-n} + a_{N,N-1}u_{N-1} + a_{N,N}u_N = b_N \end{cases} \quad (6)$$

- $N = n^2$ = number of unknowns.
- $A = (a_{i,j})$ is an $N \times N$ square matrix.
- A is a symmetric positive definite matrix (Mixed FEM).
- We used routine 'dposv' of LAPACK to solve the linear system.

Numerical Results

- Domain divided into square cells ($n \times n$).

Number of cells in each direction	n=20		n=50		n=100	
Domain	Full	Reduced	Full	Reduced	Full	Reduced
CPU(sec)	0.0089	0.0020	1.1098	0.1439	64.0162	8.1327
Flux	0.1577	0.1585	0.1568	0.1569	0.1564	0.1564
Condition Number: C_n	2390	2369	14912	14797	59761	59309

- CPU time is reduced by approximately a factor of 8.
- The flux is the same in the two cases.
- The condition number is conserved.

- The colors show the iso-values of the unknown u (Temperature).
- The arrows represent the heat flux which is related to the gradient of u.
- We obtain exactly the same results.

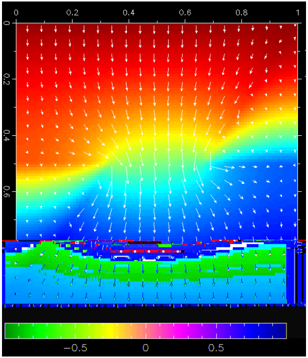


Figure: Full dense domain with mesh of size 100×100.

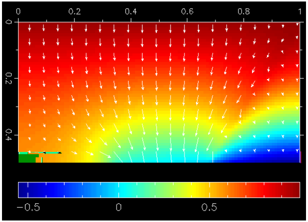


Figure: Reduced dense domain with mesh of size 100×50.

Conclusion and Future Work

- Point-symmetry problem is equivalent to a periodic radial symmetry problem with an angle $= 180^\circ$.

$\rightarrow$ We can solve any radial symmetry problem (for example 120° or any angle which is a divisor of 360°).

- The work presented in 2D can be extended to 3D.
- Many papers and books about symmetry are found in literature but very few are found about writing boundary conditions rigorously.
- A scientific paper is written and ready to be submitted.

Thank you

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